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Explicit global strong solution for 3-D compressible Navier-Stokes equations

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Abstract. In the limit of large Mach and Reynolds numbers, an exact global strong explicit analytical solution to the unsteady three-dimensional (3D) compressible Navier–Stokes (CNS) equations in unbounded space for arbitrary large smooth initial data with finite energy is obtained. The known exact explicit vortex solution in Eulerian variables for the 3D Hopf equations (S. Chefranov, 1991) is essentially used for this purpose. Our new solution to the 3D CNS equations provides a counterexample to the well-known proofs of finite-time blowup for 3D CNS solutions with large initial data, and therefore the necessity of the incompressibility condition in the formulation of the Clay Institute Navier–Stokes problem (2000) should be revisited.

1. Introduction

There is a widespread view that it is impossible to obtain an explicit analytical classical (strong) vortex solution for the nonlinear unsteady three-dimensional (3D) compressible Navier-Stokes equations (CNS) in an unbounded space using known mathematical methods[1]-[6]. The same also applies to the idealized approximation of an incompressible medium corresponding to an infinite propagation velocity of perturbations. Although it is known [5], [6] that the CNS equations are applicable, except in cases of highly rarefied media in the presence of dissociation and condensation effects [6], for describing shock-wave layers, they are excluded from consideration in the well-known formulation of the Millennium Problem by the Clay Institute (2000) [3], [4].

Because of the widespread a priori assumption that direct analytical solutions of the nonlinear Navier-Stokes equations cannot be obtained, the Clay formulation [3] restricts consideration to the incompressible approximation and focuses on proving the existence or non-existence of global solutions for arbitrary smooth finite-energy initial data in unbounded three-dimensional space.

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In addition, there are known doubts regarding the possibility of describing the effects of vortex turbulence intermittency and the Onsager dissipative anomaly problem on the basis of globally smooth strong solutions of the Navier-Stokes equations, even if such solutions exist[1], [2]. It is also believed that Clay's problem for the NavierStokes equations is an idealized mathematical version of the physically significant question of the applicability of these equations to real fluids, which creates a gap between such mathematical proofs and the physical understanding of the turbulence phenomenon[1], [2].
In our paper, these dominant views are revisited on the basis of the exact explicit analytical strong solution of the Cauchy problem obtained back in 1991 for arbitrary smooth initial data with finite energy in unbounded space for the vortex 3D Helmholtz equation in the compressible case [7]. This solution corresponds to a general class of inertial motion with zero acceleration (for example, in the cases of geostrophic or cyclostrophic balance), described by the 3D Riemann-Hopf equations. In [7], an exact explicit solution to these equations, even for the case of arbitrary spatial dimension n-D, is obtained directly in Eulerian variables using the generalized functional theory approach [8]. Global regularization of this solution is achieved by taking into account finite external Rayleigh friction or an arbitrarily small nonzero effective viscosity modeled by a random "white-noise" velocity field and the Furutsu-Novikov theory [9], [10].

In the theory developed in [11], [12], based on the exact solution [7] for the 3D compressible Navier-Stokes equations, a simpler approach has been proposed than that developed in [13]-[15] to obtain a closed description of the velocity field (see also [6]). In [11], [12], the exclusion of the dependence on density and pressure in the closed description of the velocity field was achieved by using, instead of the equation of state, an alternative description in which the pressure is directly proportional to the divergence of the velocity field [5], [16] (see (81.4) in [5], (64.5) in [6], and (3.4.8) in [16]). It is also shown in [11], [12] that such a representation for pressure follows from a sufficient positivity condition for the growth rate of integral entropy in nonequilibrium systems [17]-[36].

In our paper, we consider an explicit analytical global strong solution of the 3D compressible Navier-Stokes equations obtained on the basis of the theory [7] using a closed description of the velocity field in the limit of large Mach and Reynolds numbers, in which the n-D Navier-Stokes equations are reduced to the n-D Riemann-Hopf equations or to the inviscid n -D Burgers equation for $\mathrm{n} = 1, 2, 3, \ldots$

The basis for revising the skepticism noted in[1], [2] regarding the possibility of solving problems of turbulence theory on the basis of exact solutions of hydrodynamic equations is essentially given in [37] (see also [38]-[44]), where the development of the theory [7] provides a solution to the Onsager dissipative anomaly. In addition, in [37], directly from the exact explicit solution of the one-dimensional compressible Euler equations, generalizing the well-known implicit Riemann solution [5], the value of the exponent in the universal -8/3law for the energy spectrum of compressible turbulence is established. This turbulent spectrum is adjacent to the Kolmogorov-Obukhov -5/3law in the small-scale region. The -8/3law finds direct confirmation in studies of cosmic plasma, where the effects of compressibility and inertia dominate over viscosity [45], [46]. The -8/3spectral law was first obtained by Sulem and Frisch (1975) [47] on the basis of dimensional and similarity considerations, by analogy with the derivation of the KolmogorovObukhov $-5/3$law. In

contrast to [47], in [37] the spectral exponent in the -8/3law for the inertial range is obtained directly from an exact solution of the Euler equations for arbitrary Mach numbers, bypassing the well-known closure problem in turbulence theory.

Indeed, this classical problem, which remains unsolved, consists in obtaining an explicit closed description of averaged nonlinear fields whose realizations satisfy the hydrodynamic equations [48]. It was formulated rigorously by Monin and Yaglom in [48] (see pages 12-14 in [48]) as the problem of obtaining arbitrary moments of nonlinear fields and their spectra without solving an infinite hierarchy of equations, which Feynman considered to be an important unsolved problem of classical physics [49].

Indeed, as noted in [49]: "Finally, there is a physical problem that is common to many fields, that is very old, and that has not been solved. It is not the problem of finding new fundamental particles, but something left over from a long time agoover a hundred years. Nobody in physics has really been able to analyze it mathematically satisfactorily in spite of its importance to the sister sciences. It is the analysis of circulating or turbulent fluids."

Thus, the exact analytical solution obtained in [7] and considered in this paper makes it possible to address the turbulence problem. Moreover, the very existence of such a global strong solution provides a counterexample to the known proofs concerning the absence of global strong solutions with large initial data for the 3D CNS equations. It seems that these known proofs may also have influenced the choice of the formulation of Clay's problem [3], [4], which is limited to consideration of the incompressible case.

In the next Section 2, the problems associated with the incompressible medium approximation are considered. In Section 3, the possibility of reducing the n-D CNS equations to n -dimensional generalizations of the Riemann-Hopf equations is discussed. Section 4 presents exact solutions of the Riemann-Hopf equations in Eulerian variables and the corresponding exact solutions of the compressible Helmholtz equations for vortex fields in the three-dimensional and two-dimensional cases. In Section 5, in the context of the predictability problem, the possibility of global regularization of these solutions is considered by taking into account homogeneous friction, and in Section 6 by introducing an extremely small but nonzero effective viscosity.

The Appendix provides a derivation of the exact solution in Eulerian variables obtained in [7] for the n -dimensional generalization of the Riemann-Hopf equations and the 3D Helmholtz vortex equations for a barotropic compressible medium. An example of an exact closed analytical description of arbitrary single-point moments $\Omega_{3(2m)} = \int d^3x\vec{\omega}^{2m}; m = 1, 2, \dots$ of a three-dimensional vortex field is also obtained, which under certain conditions makes it possible to establish a relationship (see (A.8) in the Appendix) that is considered to be an important characteristic of pronounced turbulence intermittency[1]. In this paper, as in [7], [47], spatial averaging is used to obtain the moments of vortex fields instead of averaging over a statistical ensemble of realizations [48].

2. Compressible and incompressible Navier-Stokes equations

The global regularity problem for the three-dimensional (3D) Navier-Stokes equations was formulated by the Clay Mathematics Institute (2000) [3], [4] only for the incompressible case as a necessary requirement. The main reason for limiting

the problem to the idealized incompressible fluid approximation in the official formulation of the Clay Institute [3] is the desire to isolate the fundamental problem of competition between the nonlinear term, describing the possibility of singularity formation (blowup) of the solution, and viscous dissipation responsible for energy dissipation. It is believed that the incompressible approximation in describing the dynamics of a real medium represents the "purest" form of this mathematical dilemma without the need to consider additional equations and physical complications that may arise when this approximation is abandoned.

In addition, a possible reason for this formulation of the Clay Institute problem is the well-known proof by Xin (1998) [50] (see also [51]-[53]) concerning the absence of global solutions to the 3D compressible Navier-Stokes equations with finite initial data and arbitrary viscosity insufficient to prevent the formation of singularities in finite time.

We show that, with a certain approach to the problem of global regular solutions of the 3D Navier-Stokes equations, the compressible case turns out, on the contrary, to be mathematically and physically simpler for resolving the above dilemma compared with the incompressible medium approximation.

Let us consider the n-D compressible Navier-Stokes equations for constant shear viscosity, volume (or second) viscosity, and heat conductivity κ [12], [48], [5] (see (1.1), (1.4) in [48] and (2.7), (2.8) in [12]):

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \frac{\eta}{\rho}\Delta u_i - \frac{1}{\rho}\frac{\partial}{\partial x_i}\left(p - \left(\varsigma + \frac{\eta}{3}\right)\frac{\partial u_l}{\partial x_l}\right) \tag{1}$$

$$i = 1, 2, \ldots, n, \qquad \eta = \text{const}, \qquad \varsigma = \text{const}.$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0 \tag{2}$$

(3)

$$\frac{\partial}{\partial t}\left(\rho\frac{\vec{u}^2}{2} + \rho\varepsilon\right) = -\frac{\partial}{\partial x_i}\left(u_i\left(\rho\left(\frac{\vec{u}^2}{2} + \varepsilon\right) + 2p - \left(\varsigma + \frac{\eta}{3}\right)\operatorname{div}\vec{u}\right) - \eta\frac{\partial(\vec{u}^2/2)}{\partial x_i} - \kappa\frac{\partial T}{\partial x_i}\right)$$

$$\Phi = \varepsilon + \frac{p}{\rho} - Ts = \Phi_0 = \text{const};$$

(4)

$$d\varepsilon = T\,ds + \frac{p}{\rho^2}d\rho; \qquad -s\,dT + \frac{dp}{\rho} = d\Phi = 0$$

$$\frac{\partial(\rho\varepsilon)}{\partial t} = T\frac{\partial(\rho s)}{\partial t} + \Phi_0\frac{\partial \rho}{\partial t};$$

$$T\frac{\partial(\rho s)}{\partial t} + \frac{\partial}{\partial x_j}(\rho s T u_j) = \eta\left(\frac{\partial u_i}{\partial x_j}\right)^2 \tag{5}$$

$$- \left(p - \left(\varsigma + \frac{\eta}{3}\right)\frac{\partial u_k}{\partial x_k}\right)\frac{\partial u_j}{\partial x_j}$$

$$p = p(\rho; \varepsilon; s) \tag{6}$$

Here, Eq. (6) represents one of the known forms of the equation of state (EOS) [5], [15]. The conservation laws for momentum (1), mass (2), and the densities of total energy and entropy (per unit volume) (3)-(5) are also presented.

The equation of state (6) is used only to close the system of equations (1)-(3). It relates the density of the medium, its pressure, and the internal energy density (per unit mass) only under the condition of local thermodynamic equilibrium. It is assumed that such equilibrium is established on time scales much shorter than the characteristic time scale associated with the macroscopic motion of the medium [5], [6], so that Eq. (6) is locally satisfied at each point in space and time.

The mechanism of singularity formation in solutions of the compressible Navier-Stokes equations (1)-(3) is usually associated with the presence of an additional nonlinear relationship between the medium density, pressure, and velocity, depending on the specific form of the equation of state (6).

As a result, in [3] it is precisely the incompressible formulation of the problem that is considered, according to which the density of the medium is assumed to be constant $\rho = \rho_0 = \text{const}$. Then, from the law of mass conservation (2), in the absence of sources and sinks in the medium, the following equation necessarily follows

$$\operatorname{div} \vec{u} = 0 \tag{7}$$

When condition (7) is satisfied, the Navier-Stokes equations (1) and the corresponding Cauchy problem in unbounded three-dimensional space take the form:

$$\begin{aligned} &\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} + \nu \Delta u_i; \nu = \eta/\rho_0 = \text{const}; \\ &u_i(\vec{x}; t=0) = u_{oi}(\vec{x}); E_0 = \frac{1}{2}\int d^3x \vec{u}_0^2 < \infty; i = 1,2,3 \end{aligned} \tag{8}$$

The system of equations (7), (8) is closed and can therefore be considered without the need for any additional equation of state (6). At the same time, the pressure in (8) no longer has its usual thermodynamic meaning, especially when viscosity is taken into account [6]. In fact, for the incompressible Navier-Stokes equations (8) considered in the formulation of the Clay problem [3], together with the incompressibility condition (7), an additional equation is necessarily derived instead of the equation of state (6), in which the pressure depends quadratically on the spatial derivatives of the velocity field. This dependence follows from (8) by applying the divergence operator to Eq. (8) and, taking condition (7) into account, is determined from the solution of the following equation:

$$\Delta p = -\rho_0 \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} \tag{9}$$

In the general n -dimensional case, the solution of Eq. (9) has the form [54] (see also (1.9') in [48] for the case $n = 3$):

$$
\begin{aligned}
& p(\vec{x};t) = \frac{\rho_0}{2\pi}\int d^2x_1 P\left(\vec{u}\left(\vec{x}_1;t\right)\right)\ln\left|\vec{x}-\vec{x}_1\right| \equiv \rho_0\tilde{P}_2(\vec{u}:\vec{u}); n=2 \\
& p(\vec{x};t) = \frac{\rho_0\Gamma(n/2)}{2\pi^{n/2}(n-2)}\int d^n x_1 \frac{P\left(\vec{u}\left(\vec{x}_1;t\right)\right)}{\left|\vec{x}-\vec{x}_1\right|^{n-2}} \equiv \rho_0\tilde{P}_n(\vec{u}:\vec{u}); n\geq 3 \qquad (10) \\
& P(\vec{u}) \equiv \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}
\end{aligned}
$$

In Eq. (10), Γ denotes the gamma function, and $\Gamma(3/2)=\sqrt{\pi}/2$ for the three-dimensional case ($n=3$).

Thus, in the incompressible case, when solving the Cauchy problem (8), the pressure representation given by Eq. (10) should be substituted into Eq. (8), from which we obtain the following closed description of the velocity field in three-dimensional space:

$$
\begin{aligned}
& \frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{\partial\tilde{P}_3(\vec{u}:\vec{u})}{\partial x_i} + \nu\Delta u_i; \nu=\eta/\rho_0=\text{const}; \qquad (11) \\
& u_i(\vec{x};t=0)=u_{oi}(\vec{x}); E_0=\frac{1}{2}\int d^3x\vec{u}_0^2<\infty; i=1,2,3
\end{aligned}
$$

As a result, in addition to the quadratic nonlinearity contained in the left-hand side of the compressible Navier-Stokes equations (1), an additional quadratic integrodifferential distributed nonlinearity appears in the right-hand side of the incompressible Navier-Stokes equations (10), arising solely from the incompressibility assumption in the form of condition (7). Obviously, this additional nonlinearity introduces a qualitative mathematical complication into the original compressible Navier-Stokes system. The physical aspect of the incompressible medium approach is also an obvious idealization, in which perturbations must propagate at an infinite speed corresponding to an infinitely large speed of sound $c\to\infty; c^2=(\partial p/\partial\rho)_S$.

When using the incompressible Navier-Stokes equations (11), consideration of the problem of obtaining a vortex-free solution described by the velocity potential in the form $u_i=\partial\Phi/\partial x_i$ is mathematically correct.

However, for any vortex solution of equations (11), it becomes necessary to satisfy an additional condition ensuring consistency of this solution with the velocity representation following from the solution of the Cauchy problem for the three-dimensional incompressible Helmholtz vortex equations obtained from (11) after applying the curl operator to Eq. (11), which has the form [14] (see (2.2.11) in [14]):

$$
\begin{aligned}
& \frac{\partial\omega_i}{\partial t} + u_j\frac{\partial\omega_i}{\partial x_j} = \omega_j\frac{\partial u_i}{\partial x_j} + \nu\Delta\omega_i \\
& \omega_i = \varepsilon_{ijl}\frac{\partial u_l}{\partial x_j} \equiv \text{rot}\,\vec{u}; \qquad (12) \\
& \varepsilon_{123}=-\varepsilon_{213}=\varepsilon_{312}=1; i=1,2,3
\end{aligned}
$$

$$
u_i(\vec{x};t) = -\frac{1}{4\pi}\int d^3x_1\frac{\varepsilon_{ijk}\left(x_j-x_{1j}\right)\omega_k\left(\vec{x}_1;t\right)}{\left(\vec{x}-\vec{x}_1\right)^3} + \frac{\partial\Phi}{\partial x_i} \qquad (13)
$$

$$
\text{div}\,\vec{u} = \Delta\Phi = 0 \qquad (14)
$$

When considering the Cauchy problem for the system (12)-(14) in unbounded space, usually only the first term in (13) is taken into account (see [48]), which leads to a more complex nonlinearity in (12) than the nonlinear term $\vec{N} = (\vec{u}\vec{\nabla})\vec{u}$ in the compressible Navier-Stokes equation (1).

Thus, in addition to the obvious complication of the type of nonlinearity in equations (11) and (12) compared with the original quadratic nonlinear term in (1), it is additionally necessary to satisfy the condition ensuring consistency of the solution of the Cauchy problem for Eq. (11) with the velocity-field representation (13) following from the solution of the Cauchy problem for Eq. (12) for the vortex field.

In [48], it is only stated that since all three components of the velocity field can be determined from the solution of equations (12) (see (1.7), (1.8) in [48]), the pressure is then determined from (10). As a result, in [48], the solution of the Navier-Stokes Cauchy problem (8) or (11) in the incompressible case is implicitly assumed to be equivalent to the solution of the Cauchy problem for the Helmholtz equations (12), without an appropriate comparison between the solution of equations (11) and representation (13).

Let us assume that such an equivalence between the solutions of the Cauchy problems in unbounded three-dimensional space for the incompressible Navier-Stokes equations (11) and the Helmholtz equations (12) indeed holds. Then one should conclude that the pressure gradient does not contribute to the acceleration of fluid particles in equations (11), since in deriving (12) the corresponding gradient term on the right-hand side of (11) vanishes after applying the curl operator. It then follows that, in the limit of vanishing shear viscosity, the three-dimensional incompressible Navier-Stokes equations become equivalent to the three-dimensional Burgers or Riemann-Hopf equations.

If, on the other hand, the velocity-field solution (13) following from the solution of the Cauchy problem for the Helmholtz equations (12) does not uniquely coincide with the solution of the Cauchy problem for the Navier-Stokes equations (11), then this may indicate that the consideration of vortex solutions of the Navier-Stokes equations (11) in unbounded space is not mathematically consistent.

However, despite the indicated problems associated with the incompressible medium approximation, which is clearly no simpler than the compressible case, the idea reflected in the formulation of the Clay problem [3] still dominates, namely that the incompressible case allows one to directly consider only the principal problem of the competition between the nonlinear term $\vec{N} = (\vec{u}\vec{\nabla})\vec{u}; |\vec{N}| \cong u^2/l; u = \left\langle \vec{u}^2 \right\rangle^{1/2}$ and the viscous term $\vec{D} \equiv \nu\Delta\vec{u}; |\vec{D}| \cong \nu u/l^2$ in the limit of large Reynolds numbers $\mathrm{Re} = |\vec{N}|/|\vec{D}| \cong ul/v \gg 1$.

This problem arises from the fact that the three-dimensional nonlinear effect of vortex-line stretching leads to strong vortex interactions in the small-scale region and to an explosive growth of enstrophy $\Omega_3 = \int d^3x\vec{\omega}^2$. This growth may become unbounded in finite time due to insufficient energy dissipation and lead to the formation of a singularity in the solution of the three-dimensional Helmholtz vortex equation (12).

3. Reduction of Navier-Stokes equations to Hopf equations

Let us show that consideration of the compressible case, described by the seemingly cumbersome system of equations (1)-(6), can in fact be reduced to the consideration of vortex dynamics generalizing equation (12).

Indeed, in the compressible case, the three-dimensional Helmholtz vortex equation is obtained from the three-dimensional compressible Navier-Stokes equations (1) (for example, in the simplest barotropic case $p = p(\rho)$) after applying the curl operator $\omega_i = \varepsilon_{ijk}\partial u_k/\partial x_j$ to them. Equation (12) is thereby generalized and takes the following well-known form (in the limit of vanishing viscosity $\eta \to 0; \varsigma \to 0$[15], [55]):

$$
\begin{aligned}
&\frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \omega_j \frac{\partial u_i}{\partial x_j} - \omega_i div\vec{u} \\
&\omega_i = \varepsilon_{ijl} \frac{\partial u_l}{\partial x_j} \equiv \text{rot}\, \vec{u}; \\
&\varepsilon_{123} = -\varepsilon_{213} = \varepsilon_{312} = 1; i = 1, 2, 3
\end{aligned} \tag{15}
$$

For the incompressible approximation, only one term $-\omega_i div\vec{u}$, which is present in equation (15), is absent from the right-hand side of the Helmholtz equation (12), while the remaining terms of equation (15) for the vortex field in the compressible case coincide with those in equation (12) for the incompressible case, provided that the vanishing-viscosity limit is also considered in (12).

Thus, in addition to the term present in the incompressible case and responsible for vortex-line stretching, the three-dimensional compressible Helmholtz vortex equations contain an additional nonlinear term $-\omega_i div\vec{u}$, which leads to an additional explosive growth of enstrophy not only in the three-dimensional case, but even in the two-dimensional case. Therefore, the problem of competition between the nonlinear and viscous terms appears even more relevant in the compressible case, since in the incompressible case, in addition to the necessary quadratic nonlinearity $(\vec{u}\vec{\nabla})\vec{u}$, there is also, as noted above, an additional artificial nonlinearity associated with the pressure field and defined through the quadratic velocity-field functional in (11).

According to the global strong solution of equations (15) obtained in [7] for arbitrarily small but nonzero effective viscosity, the conclusions of [50]-[53] regarding the absence of global solutions for the compressible Navier-Stokes equations require reconsideration in the limit of large Mach numbers $Ma = u/c; c^2 = (\partial p/\partial \rho)_S$, where c is the local speed of sound at constant entropy. Indeed, since the conclusions obtained in [50]-[53] are assumed to be valid for arbitrary Mach numbers, the existence of a global strong solution established in [7] provides a counterexample to these conclusions.

The vortex solution was obtained in [7] in Eulerian variables for the three-dimensional Riemann-Hopf equations, for which the Cauchy problem in unbounded space with an arbitrary integer number of dimensions has the form:

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=0; i=1,2,3,..,n; \tag{16}$$
$$u_i(\vec{x};t=0)=u_{0i}(\vec{x})\in C^2; I_E=\int d^nx\vec{u}_0^2<\infty$$

It is easy to see that after applying the curl operator to Eqs. (16) in the three-dimensional case $n = 3$, the exact three-dimensional Helmholtz equations (15) are obtained. Therefore, the exact solution of the Cauchy problem (16) made it possible in [7] to obtain an exact analytical solution of the Cauchy problem for the three-dimensional compressible Helmholtz equations (15).

The solutions of equations (16) and (15) obtained in [7], as well as the possibility of their global regularization due to the additional inclusion of linear Rayleigh friction or effective viscosity in (16), are discussed in more detail in Sections 4-6. Here we also note the connection between equations (16) and other well-known equations, whose exact analytical solutions become possible due to the solution of equations (16), obtained for the first time in [7] directly in Eulerian variables.

Equation (16), together with the continuity equation (2), coincides with the three-dimensional compressible Euler equations in the limit of large Mach numbers, when the acceleration of fluid particles caused by the pressure gradient becomes extremely small, since for the barotropic case $\vec{P}\equiv\vec{\nabla}p/\rho=c^2\vec{\nabla}\rho/\rho; |\vec{P}|\cong c^2/l$ and $\vec{N}\equiv(\vec{u}\vec{\nabla})\vec{u}; |\vec{N}|\cong u^2/l$ one may obtain the relation $|\vec{P}|/|\vec{N}|\cong O\left(1/Ma^2\right)\to 0, Ma=u/c\to\infty$ according to the known estimates [56], [39].

Euler considered the case of a balance between the density of an external force and the pressure gradient, in which fluid particles have zero acceleration and equations (16) are satisfied. In this case, he presented only the simplest partial solutions corresponding either to complete absence of motion of the fluid particles or to motion with a constant velocity identical for all fluid particles [57] (see (37)-(39) in [57]).

In this regard, the exact solution of equations (16) obtained in Eulerian variables in [7] represents a generalization of these Eulerian solutions to arbitrary initial conditions corresponding to the possibility of motion of different fluid particles with different constant velocities.

In addition, the exact solution of equations (16) makes it possible, for arbitrary initial data, to obtain an exact solution of the dispersionless Kadomtsev-Petviashvili (dKP) equation, which is a universal equation describing the propagation of nonlinear waves in acoustics, plasma physics, and hydrodynamics [58]:

$$\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x_1}\right)=-\frac{c}{2}\sum_{i=2}^{n}\frac{\partial^2 u_1}{\partial x_i^2} \tag{17}$$

Until now, only solution (17) corresponding to extremely small initial amplitudes of nonlinear waves has been known, according to which finite-time nonlinear wave collapse can be avoided only in the case of spatial dimension $n > 3$[58]. In [58], the solution of the Cauchy problem for equation (17) is reduced to the solution of the Hopf equation (16) in the one-dimensional case (see Eq. (19) in [58]).

In [7], the exact analytical solution of the Hopf equations (16) is obtained for arbitrary smooth initial data, which makes it possible to derive not only vortex solutions of the Helmholtz equations (15), but also potential velocity fields. This

makes it possible to obtain exact analytical solutions for the Sivashinsky equation [59], [60] and the Kardar-Parisi-Zhang (KPZ) equation [61].

The Sivashinsky equation describes the propagation of a flame front generated by a self-sustaining exothermic chemical reaction, for example, in the following simplified form:

$$\frac{\partial f}{\partial t} - \frac{U_S}{2}(\vec{\nabla} f)^2 = \gamma_0 f \tag{18}$$

For the case $u_i = -U_S \partial f / \partial x_i$ corresponding to Eq. (16), it is possible to reduce Eq. (18) to the form of Eq. (16) if the term $\gamma_0 u_i$ is added to the RHS of Eq. (16).

The KPZ equation universally describes processes occurring at the boundary between two media in various physical, mathematical, and biological problems and has a form similar to equation (18):

$$\frac{\partial h}{\partial t} - \frac{\lambda}{2}(\vec{\nabla} h)^2 = \nu \Delta h + \xi(\vec{x}; t) \tag{19}$$

The RHS of Eq. (19), in addition to the structure of Eq. (16), also includes viscous dissipation (the first term) and a random force.

4. Exact solution of n-D Hopf equations in Euler variables

The exact solution of the Cauchy problem for equations (16) in unbounded n dimensional space, obtained in Eulerian variables in [7], has the following form (see Appendix):

$$u_i(\vec{x}; t) = \int d^n \xi u_{0i}(\vec{\xi}) \det \hat{A}(\vec{\xi}; t) \delta \left(\vec{\xi} - \vec{x} + t \vec{u}_0(\vec{\xi}) \right)$$

$$\hat{A}(\vec{x}; t) \equiv A_{ij} = \delta_{ij} + t \frac{\partial u_{0i}}{\partial x_j}; \det \hat{A}\left(\vec{x}; t \in [0, t_{nC})\right) > 0 \tag{20}$$

Solution (20) for the velocity field in n -dimensional space also provides the exact solution of the Cauchy problem for the density distribution following from the continuity equation (2):

$$\rho(\vec{x}; t) = \int d^n \xi \rho_0(\vec{\xi}) \delta \left(\vec{\xi} - \vec{x} + t \vec{u}_0(\vec{\xi}) \right); 0 \le t < t_{nC} \tag{21}$$

$$\rho(\vec{x}; t = 0) = \rho_0(\vec{x}) \in C^1; I_\rho = \int d^n x \rho_0(x) < \infty$$

The exact solution of the Cauchy problem for the vortex-field distribution, obtained from the three-dimensional Helmholtz equations (15) for the velocity field (20), has the following form:

$$\omega_i(\vec{x}; t) = \int d^3 \xi \left(\omega_{0i}(\vec{\xi}) + t \omega_{0j} \frac{\partial u_{0i}}{\partial \xi_j} \right) \delta \left(\vec{\xi} - \vec{x} + t \vec{u}_0(\vec{\xi}) \right); i = 1, 2, 3 \tag{22}$$

$$0 \le t < t_{3C}$$

Similarly, in the two-dimensional case, the exact solution for the vortex-field distribution has a form similar to the density distribution (21):

$$\omega(\vec{x};t)=\int d^2\xi\omega_0(\vec{\xi})\delta\left(\vec{\xi}-\vec{x}+t\vec{u}_0(\vec{\xi})\right) \tag{23}$$
$$\omega_0(\vec{x})=\varepsilon_{ij}\partial u_{0j}/\partial x_i\in C^1;0\le t<t_{2C}$$

Solutions (20)-(23), obtained for an ideal compressible medium without taking into account flow-energy dissipation due to shear viscosity or friction, retain smoothness only over a finite time interval. The value of this interval is determined by the minimum evolution time of these solutions at which the condition $\det\hat{A}>0$ noted in (20) is satisfied. Depending on the spatial dimension, this value is determined from the following conditions for the one-dimensional, two-dimensional, and three-dimensional cases:

$$n=1:\det\hat{A}=1+tdu_{01}/dx_1=0$$
$$t_{1C}=\frac{1}{\max_{x_1}|du_{01}/dx_1|};\det\hat{A}>0\Rightarrow 0\le t<t_{C1} \tag{24}$$

$$n=2:\det\hat{A}=1+t_2\,\mathrm{div}\,\vec{u}_0+t_2^2U_{012}=0;t_{2C}=\min_{x_1;x_2}t_2(\vec{x})$$
$$n=3:\det\hat{A}=1+t_3\,\mathrm{div}\,\vec{u}_0+t_3^2\left(U_{012}+U_{013}+U_{023}\right)+t_3^3\det\left(\frac{\partial u_{0i}}{\partial x_j}\right) \tag{25}$$
$$U_{012}=\frac{\partial u_{01}}{\partial x_1}\frac{\partial u_{02}}{\partial x_2}-\frac{\partial u_{02}}{\partial x_1}\frac{\partial u_{01}}{\partial x_2};t_{3C}=\min_{x_1;x_2;x_3}t_3(\vec{x})$$

For example, in the two-dimensional case with zero divergence of the initial velocity field, according to (25) we obtain the following estimate for the minimum lifetime of the smooth solutions (20), (21), and (23):

$$u_{0i}=\varepsilon_{ij}\partial\psi_0/\partial x_j;\psi_0=a\sqrt{L_1L_2}\exp\left(-\frac{x_1^2}{L_1^2}-\frac{x_2^2}{L_2^2}\right)>0; \tag{26}$$
$$t_{2C}=\frac{e\sqrt{L_1L_2}}{2a}$$

Solutions (20)–(23), over the finite time interval determined by the condition $\det\hat{A}>0$, are strong (classical) solutions of equations (2), (15), and (16). This can be verified by direct substitution of these solutions into the corresponding equations, taking into account the following relations associated with the known properties of generalized Dirac delta functions δ [8], [54], which are contained in all solutions presented in [7], [11], [12]:

$$\delta\left(\vec{\xi}_2-\vec{\xi}_1+t\left(\vec{u}_0\left(\vec{\xi}_2\right)-\vec{u}_0\left(\vec{\xi}_1\right)\right)\right)\equiv\frac{\delta\left(\vec{\xi}_2-\vec{\xi}_1\right)}{\det\hat{A}\left(\vec{\xi}_1;t\right)} \tag{27}$$

$$\frac{\partial\delta\left(\vec{\xi}-\vec{x}+t\vec{u}_0\right)}{\partial x_i}\equiv-A_{ji}^{-1}(\vec{\xi};t)\frac{\partial\delta\left(\vec{\xi}-\vec{x}+t\vec{u}_0(\vec{\xi})\right)}{\partial\xi_j} \tag{28}$$
$$A_{li}A_{im}^{-1}=\delta_{lm}$$

In addition, to verify these solutions by substituting them into the equations, it is also necessary to use the following relations [7], [11], [12]:

$$\frac{\partial \det \hat{A}}{\partial t} \equiv \frac{\partial u_{0m}}{\partial \xi_k} A^{-1}_{km} \det \hat{A} \tag{29}$$

$$\frac{\partial}{\partial \xi_k}\left(A^{-1}_{km} \det \hat{A}\right) = 0 \tag{30}$$

In [7] (and in the next paragraph), the possibility of regularization of solutions (20)-(23) is shown, in which all these strong solutions become globally smooth in time, for example, by taking into account linear friction $-\mu\vec{u}$ with a sufficiently large coefficient $\mu > \mu_{th} = t^{-1}_{nC}$ on the right-hand side of (16).

The solutions are regularized even more effectively by introducing an arbitrarily small but nonzero effective viscosity modeled by a homogeneous random velocity field of the "white-noise" type through the replacement

$$u_j \frac{\partial u_i}{\partial x_j} \to \left(u_j + V_j(t)\frac{\partial u_i}{\partial xj}\right)$$

in (16), together with the Furutsu-Novikov relation [9],[10]

$$\left\langle V_j(t)\frac{\partial u_i}{\partial x_j}\right\rangle = -\nu_E \frac{\partial^2 \langle u_i\rangle}{\partial x_j^2}$$

5. Solving the problems of global regularization and predictability when the finite threshold friction is taken into account.

The solution proposed in [7] and considered below for the problem of global regularization of the Helmholtz equations may have an important aspect in connection with the well-known fundamental and applied problem of predictability of hydrometeorological fields and their statistical characteristics arising in the use of numerical solutions of the Navier-Stokes equations (1) or (8), (11) [62], [63].

The problem of predictability, or the problem of determining the time interval over which a sufficiently reliable forecast is possible, arises from the nonlinearity of the equations and the sensitivity of their solutions to even small unavoidable errors in the initial data. For example, for solutions (20)-(23) that lose smoothness in finite time $t = t_{nC}$, the predictability limit is determined precisely by this time value, which itself depends significantly on the initial data, as follows from (24)-(26).

The mathematical aspect of the predictability problem, related to the need to clarify not only the existence but also the uniqueness of global solutions, was formulated by Ladyzhenskaya (2003) [62]. For the three-dimensional Navier-Stokes equations (8), an alternative to the formulation of the problem proposed by the Clay Institute [3] is given in [62] in the following form:
"Do the Navier-Stokes equations, together with the initial and boundary conditions, provide a deterministic description of fluid dynamics, or do they not?"

One of the well-known key factors limiting the predictability of the statistical characteristics of turbulent flows is the clipping procedure typical of numerical predictive schemes using the Navier-Stokes equations (1) and (8) at a certain minimum computational grid-cell size $l \geq l_{\min}$ [63].

In this case, an additional effect of numerical viscosity or linear velocity friction $-\mu\vec{u}$ arises, with the coefficient $\mu = \nu/l^2_{\min}$.

This follows from the truncation procedure, in which the Laplacian contribution is estimated with finite precision as $\Delta \simeq l^{-2}_{\min}$, which corresponds to the substitution $\vec{D} \equiv \nu\Delta\vec{u} \to \vec{L} = -\mu\vec{u}$.

Therefore, we will consider the possibility of such a linear dissipation rate $\vec{L}$ competing with the nonlinearity $\vec{N} \equiv (\vec{u}\vec{\nabla})\vec{u}$ in equation (16) by introducing the following modification of Eq. (16):

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\mu u_i; i = 1, 2, 3, .., n; \mu = \frac{\nu}{l^2_{\min}} \tag{31}$$
$$u_i(\vec{x}; t = 0) = u_{0i}(\vec{x}) \in C^2; I_E = \int d^n x\vec{u}^2_0 < \infty$$

An exact solution of the Cauchy problem (31) has the following form:

$$u_i(\vec{x}; t) = \int d^n\xi u_{0i}(\vec{\xi}) \det\hat{A}(\vec{\xi}; \tau(t))\delta\left(\vec{\xi} - \vec{x} + \tau(t)\vec{u}_0(\vec{\xi})\right)$$
$$\hat{A}(\vec{x}; \tau(t)) \equiv A_{ij} = \delta_{ij} + \tau(t)\frac{\partial u_{0i}}{\partial x_j}; \det\hat{A}\left(\vec{\xi}; \tau(t) \in [0; t_{nC})\right) > 0 \tag{32}$$
$$\tau(t) = \frac{1 - \exp(-t\mu)}{\mu} < t_{nC} \Rightarrow 0 \le t < \frac{1}{\mu}\ln\left(\frac{1}{1 - \mu t_{nC}}\right), \mu < \mu_{th} = t^{-1}_{nC}$$
$$\forall t \in [0, \infty) if \mu > \mu_{th} = 1/t_{nC}$$

Thus, according to the exact strong solution of equations (31) in the explicit analytical form (32), for a superthreshold value of the linear friction coefficient $\mu > \mu_{th} = 1/t_{nC}$, this solution becomes globally smooth in time.

Similarly, for the density equation (2) and the vortex equations (15) (after adding the term $-\mu\vec{\omega}$ to the right-hand side of (15)), the corresponding strong global solution is obtained from solutions (21)-(23) by replacing $t \to \tau(t)$, where $\tau(t)$ is defined in (32), and the value t_{nC} is determined from solutions (24) and (25).

As a result, the obtained strong global solution (32) makes it possible to understand the underlying cause of the previously unresolved predictability problem associated with the use of the small-scale truncation procedure in numerical predictive schemes based on the Navier-Stokes equations (1) or (8).

Indeed, until now, the choice of the minimum grid size $l_{\min}$ in numerical calculations has been made without the necessary consideration of the initial data (defined below in condition (33)), which must be taken into account according to the condition $\mu > \mu_{th} = t^{-1}_{nC}$ for global regularization of the solution in time obtained in (32).

Under this condition, even relatively simple linear friction can compete with nonlinear effects. In the three-dimensional case, from the condition for the existence of the strong global solution (32), we obtain the following restriction on the minimum grid size, which provides a positive resolution of the predictability problem:

$$\mu = \frac{\nu}{l_{\min}^2} > \mu_{th} = t_{3C}^{-1} \Rightarrow l_{\min} < \sqrt{\nu t_{3C}} \cong l\,\mathrm{Re}^{-1/2}, \mathrm{Re} = ul/\nu;$$

$$t_{3C} = t_{3C}\left(\mathrm{div}\,\vec{u}_0; \det \hat{U}_{012}; \det \hat{U}_{013}; \det \hat{U}_{023}; \det \hat{U}_0\right); \tag{33}$$

$$\hat{U}_0 = \frac{\partial u_{0i}}{\partial x_j}; \hat{U}_{0\alpha\beta} = \frac{\partial u_{0\alpha}}{\partial x_\alpha}\frac{\partial u_\beta}{\partial x_\beta} - \frac{\partial u_{0\alpha}}{\partial x_\beta}\frac{\partial u_{0\beta}}{\partial x_\alpha}$$

Equation (33) presents the result of solving problem (25) for $n = 3$, when the minimum collapse time t_{3C} of a nonlinear wave is determined. This quantity is a functional of the expressions indicated in parentheses containing spatial derivatives of the initial velocity field.

At the same time, Eq. (33) does not imply the usual summation convention for repeated indices $\alpha = 1, 2, 3$ and $\beta = 1, 2, 3$, for which the condition $\alpha \neq \beta$ must be satisfied.

Thus, an example of a positive solution is obtained for the principal problem of competition between nonlinear and dissipative terms discussed in the Introduction and underlying the well-known formulation of the Clay Institute problem restricted to the incompressible approximation [3], [4].

It is significant that the linear velocity friction in (31), unlike the transition to a higher-order operator through the replacement $\nu\Delta\vec{u} \to \nu\Delta^\alpha\vec{u}, \alpha \geq 5/4$, in (8) (see [64]-[66]), does not artificially enhance dissipation. On the contrary, it does not account for the important effect of energy dissipation in the region of extremely small scales $l \ll l_{\min}$, which is described by the usual dissipative term $\nu\Delta\vec{u}$.

The result of further modeling of energy dissipation caused by the structurally similar replacement $\nu\Delta\vec{u} \to \nu_E\Delta\vec{u}$ shows that even taking into account an arbitrarily small but nonzero effective viscosity $\nu_E > 0$ is sufficient to ensure global regularization in time for the corresponding modifications of solutions (20)-(23).

6. Regularization due to any non-zero effective viscosity.

Indeed, in [7], global regularization of solutions (20)-(23) is achieved by additionally taking into account effective viscosity modeled by "white noise" in obtaining an exact analytical solution in Eulerian variables for the three-dimensional Hopf equations:

$$\frac{\partial u_i}{\partial t} + (u_j + V_j(t))\frac{\partial u_i}{\partial x_j} = 0;$$

$$\langle \vec{V} \rangle = 0; \langle V_i(t_1) V_j(t_2) \rangle = 2\nu_E \delta_{ij} \delta(t_1 - t_2); \tag{34}$$

$$\left\langle V_j(t)\frac{\partial u_i}{\partial x_j} \right\rangle = -\nu_E \Delta u_i$$

In (34), a random Gaussian velocity field $\vec{V}$, delta-correlated in time, is introduced. It is proposed as a model for the effect of effective shear viscosity ν_E on the statistically averaged velocity $\langle\vec{u}\rangle$, density $\langle\rho\rangle$, and vorticity $\langle\vec{\omega}\rangle$, which, as a result of averaging over the random velocity field, become globally smooth in contrast to the initial representations of these fields in (20)-(23).

Indeed, equations (34) admit an exact solution whose structure coincides with that of solution (20) if the argument of the delta function in (20) is replaced by:

$$\vec{x} \to \vec{x} - \vec{B}(t);$$

$$B_i = \int_0^t dt_1 V_i\left(t_1\right); \langle B_i \rangle = 0; \left\langle \vec{B}^2 \right\rangle = 2\nu_E n t; i = 1, 2, \ldots, n \tag{35}$$

$$P(\vec{B}) = \frac{1}{\left(2\left\langle \vec{B}^2 \right\rangle^{1/2}\right)^n} \exp\left(-\frac{\vec{B}^2}{2\left\langle \vec{B}^2 \right\rangle}\right)$$

Similarly, modifications of the exact solutions for the density and vortex fields are obtained from the representations of these fields in (21)-(23).

We statistically average the resulting modifications of solutions (20)-(23) over a random field $\vec{B}$, which also has a Gaussian probability distribution $P(\vec{B})$, given in (35) for the case of an arbitrary integer spatial dimension.

As a result, we obtain the following exact strong global solutions for the velocity and density fields in a space of arbitrary dimension n :

$$\langle u_i(\vec{x};t)\rangle = \frac{1}{\left(2\sqrt{n\pi t\nu_E}\right)^n} \int d^n\xi u_{0i}(\vec{\xi}) \det \hat{A}(\vec{\xi};t) \exp\left(-\frac{\left(\vec{x} - \vec{\xi} - t\vec{u}_0(\vec{\xi})\right)^2}{4\nu_E n t}\right)$$

(36)

$$i = 1, 2, 3, \ldots, n$$

$$\vec{u}(\vec{x}; t = 0) = \vec{u}_0(\vec{x}); I_E = \int d^n x \vec{u}_0^2 < \infty; \vec{u}_0 \in C^1$$

(37)

$$\langle \rho(\vec{x};t)\rangle = \frac{1}{\left(2\sqrt{n\pi t\nu_E}\right)^n} \int d^n\xi \rho_0(\vec{\xi}) \exp\left(-\frac{\left(\vec{x} - \vec{\xi} - t\vec{u}_0(\vec{\xi})\right)^2}{4\nu_E n t}\right)$$

Similarly, we obtain the following exact strong globally smooth solutions for the modified compressible Helmholtz equations (15) (in this case, in Eq. (15) it is necessary to replace

$$(\vec{u}\vec{\nabla})\vec{\omega} \to ((\vec{u} + \vec{V}(t))\vec{\nabla})\vec{\omega}$$

in the three-dimensional and two-dimensional cases, respectively:

$$\langle \omega_i(\vec{x};t)\rangle = \frac{1}{\left(2\sqrt{3\pi t\nu_E}\right)^3} \int d^3\xi \left(\omega_{0i}(\vec{\xi}) + t\omega_{0j}\frac{\partial u_{0i}}{\partial \xi_j}\right)$$

$$\times \exp\left(-\frac{\left(\vec{x} - \vec{\xi} - t\vec{u}_0(\vec{\xi})\right)^2}{12\nu_E t}\right); \quad i = 1, 2, 3 \tag{38}$$

$$\vec{\omega} = \operatorname{rot} \vec{u}; \quad \vec{\omega}_0 = \operatorname{rot} \vec{u}_0.$$

$$\langle \omega(\vec{x};t)\rangle = \frac{1}{\left(2\sqrt{2\pi t\nu_E}\right)^2}\int d^2\xi\omega_0(\vec{\xi})\exp\left(-\frac{\left(\vec{x}-\vec{\xi}-t\vec{u}_0(\vec{\xi})\right)^2}{8\nu_E t}\right) \tag{39}$$

7. Conclusions

Thus, in the limit of large Mach and Reynolds numbers, an example of an exact strong global solution for the three-dimensional vortex field is obtained in (36) and (38) for the three-dimensional compressible Navier-Stokes equations.

This makes it possible to obtain a closed description of arbitrary one-point and multipoint moments and spectra of the velocity and vortex fields and, in particular, to solve the problem of the dissipative Onsager anomaly, which is of central importance in turbulence theory[1].

Indeed, in [44], based on solutions (36) and (38), the following dependence of enstrophy on the Reynolds number was obtained:

$$\Omega_3(\mathrm{Re}) = \int d^3x\vec{\omega}^2 \cong \mathrm{Re}^{1/2}$$

according to which, when describing turbulence on the basis of the three-dimensional equations (1), the average rate of turbulent-energy dissipation nevertheless tends to zero in the low-viscosity limit:

$$\langle\varepsilon\rangle = v\Omega_3 \approx \mathrm{Re}^{-1/2} \to 0, \mathrm{Re} = ul/v \to \infty$$

According to the accepted terminology[1], this corresponds to the so-called weak anomaly, in contrast to the strong anomaly,

$$\langle\varepsilon\rangle \to \langle\varepsilon_0\rangle = \text{ const}, \ \mathrm{Re}\to\infty,$$

which corresponds to the well-known Kolmogorov hypothesis, or the "zeroth law" of turbulence[1].

In [44], based on the exact solution (36), the possibility of realizing this Kolmogorov hypothesis was nevertheless established for the case of an arbitrary integer spatial dimension. However, this turns out to be possible only if the turbulence phenomenon is described using the Navier-Stokes equations (1) in a six-dimensional space $n = 6$.

For all other dimensions, the average rate of turbulent-energy dissipation tends to zero in the limit $\mathrm{Re}\to\infty$. Therefore, the question arises as to the interpretation of observational data according to which, for sufficiently large Reynolds numbers $\mathrm{Re} > 10^4$, the turbulent-energy dissipation rate tends to a finite nonzero limit[1].

Based on the conditions for applicability of the Kolmogorov hypothesis obtained in [44], an adequate description of the turbulence phenomenon is possible only on the basis of the Navier-Stokes equations (1) formulated in a six-dimensional Cartesian space. This means that, when formulating the hydrodynamic equations in this way, each point in space occupied by a turbulent medium must be characterized not by three, but by six mutually orthogonal vectors describing all necessary degrees of freedom of this vortex turbulent state.

In addition, the examples of strong global solutions of the three-dimensional compressible Navier-Stokes equations (1) considered in (32) and (36), (38) may

serve as counterexamples to the conclusions obtained in [50]-[53] concerning the inevitability of finite-time loss of smoothness for arbitrary Mach and Reynolds numbers when the smooth initial data are finite rather than infinitesimally small in amplitude.

Indeed, solutions (32) and (36), (38) are obtained precisely for finite smooth initial velocity fields having a finite energy integral in unbounded space. The uniqueness of the obtained exact analytical solutions (32) and (36), (38) follows from the existence of a common majorant for the solutions themselves and all their derivatives for given finite initial data in $\mathbb{R}^n$, according to Ladyzhenskaya's criterion [62] (see (4.17) in [62]).

Our conclusion regarding the possibility of regularization of solutions to the compressible Navier-Stokes equations in the three-dimensional case suggests that viscous dissipation can effectively compensate not only for the stretching of vortex filaments, but also for the compressibility-specific mechanism of vorticity generation associated with the presence of sources and sinks. This, in turn, suggests that a similar regularization mechanism may also operate in the incompressible case, where the only mechanism of three-dimensional energy concentration is associated with the stretching of vortex filaments [70].

Appendix

1. Let us derive the explicit form (20) for the solution of the n-dimensional Hopf equations (16) in Eulerian variables. Usually, only the representation of solution (16) in Lagrangian variables is widely used [67]. In [7], an exact explicit solution of the Hopf equations for an arbitrary number of spatial dimensions was obtained for the first time in Eulerian variables in the form (20).

Up to now, only an implicit dependence on the initial data for the solution of the Cauchy problem for the Hopf equations (16) in unbounded space has been known. This represents the n -dimensional generalization (in the limit of large Mach numbers) of the well-known implicit Riemann solution (1860) obtained for the one-dimensional compressible Euler equations [5]:

$$u_i(\vec{x};t) = u_{0i}(\vec{x} - t\vec{u}(\vec{x};t)) \tag{A.1}$$
$$u_i(\vec{x};t=0) = u_{0i}(\vec{x}); i = 1,2,3,..,n$$

Consider the Fourier transform for the solution (A.1):

$$\tilde{u}_i(\vec{k};t) = \int d^n x u_{0i}(\vec{x} - t\vec{u}(\vec{x};t)) \exp(i\vec{k}\vec{x}) \tag{A.2}$$

Let's perform the following variable substitution in (A.2):

$$\vec{\xi} = \vec{x} - t\vec{u}(\vec{x};t) \equiv \vec{x} - t\vec{u}_0(\vec{\xi}) \tag{A.3}$$

As a result of the variable transformation in (A.2) given by (A.3), we obtain the following explicit dependence of the Fourier transform of the exact solution of the n-dimensional Hopf equations (16) on the initial data:

$$\tilde{u}_i(\vec{k};t) = \int d^n\xi u_{0i}(\vec{\xi}) \det \hat{A}(\vec{\xi};t) \exp\left(i\vec{k}\left(\vec{\xi} + t\vec{u}_0(\vec{\xi})\right)\right)$$

$$\hat{A} \equiv A_{ij} = \delta_{ij} + t\frac{\partial u_{oi}(\vec{\xi})}{\partial \xi_j} \tag{A.4}$$

Only in the one-dimensional case was the explicit spectral form (A.4) of the representation for the Riemann solution previously obtained in [68], [69].

By taking into account the known representation $\delta(\vec{x}) = (2\pi)^{-n}\int d^n k \exp(-i\vec{k}\vec{x})$ for the Dirac-Heaviside delta function [8], [54], from (A.4), after the inverse Fourier transform $u_i(\vec{x};t) = (2\pi)^{-n}\int d^n k\tilde{u}_i(\vec{k};t)\exp(-i\vec{k}\vec{x})$, an exact explicit solution is obtained in the form represented in (20):

$$u_i(\vec{x};t) = \int d^n\xi u_{0i}(\vec{\xi}) \det \hat{A}(\vec{\xi};t)\delta\left(\vec{\xi} - \vec{x} + t\vec{u}_0(\vec{\xi})\right) \tag{A.5}$$

2. An exact solution of the three-dimensional Helmholtz equations (15) in the form (22) is obtained by applying the relation $\omega_i = \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}$ to solution (A.5) for the velocity field in three-dimensional space and taking relations (27)-(30) into account. Similarly, solution (23) for the vortex field in the two-dimensional case is obtained.

Solution (22) indeed satisfies the three-dimensional Helmholtz equations (15), as can be verified by directly substituting the representations for the velocity field (20) and the vortex field (22) into (15) and using relations (27)-(30) (see the Appendices in [11] and [12]).

Using the exact solution for the vortex field in the explicit form (22), together with relations (27), (28), it is possible to obtain the following exact closed analytical solution for arbitrary single-point moments and the corresponding spectra describing three-dimensional vortex turbulence [7], [11], [12].

In particular, for arbitrary even single-point moments of the vortex field, we obtain the following exact closed representation:

$$\Omega_{3(2m)}(t) = \int d^3x \frac{\left(\omega_{0i} + t\omega_{0j}\partial u_{0i}/\partial x_j\right)^{2m}}{\det{}^{2m-1}\hat{A}(\vec{x};t)} \cong O\left(1/\left(1 - t/t_{3C}\right)^{2m-1}\right) \tag{A.6}$$
$$0 \le t < t_{3C}$$

In (A.6), for the minimum singularity time t_{3C}, representation (25) must be used.

Regularization of solution (A.6) by taking homogeneous friction into account under condition (33) leads to the substitution $t \to \tau(t) = \left(1 - e^{-\mu t}\right)/\mu$ in (A.6). In the limit of large times $\mu t \gg 1$, we obtain from (A.6) the estimate

$$\Omega_{3(2m)} \cong \left(1 - \mu_{th}/\mu\right)^{-(2m-1)} \tag{A.7}$$
$$\mu_{th} = t_{3C}^{-1}$$

From (A.7), in the limit $\mu \to \mu_{th}, \mu > \mu_{th}$, it is possible to obtain the following estimate:

$$\frac{\Omega_{3(4m)}}{\Omega^2_{3(2m)}} \cong \frac{1}{1-\mu_{th}/\mu} \gg 1 \tag{A.8}$$
$$\mu \to \mu_{th}$$

This relationship between the moments of the vortex field characterizes strong turbulent intermittency with rare but intense vortex bursts [64].

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